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Research Statement

I am interested in geometry and topology, particularly through visualization, computation and application. In my Ph.D. dissertation, I studied a hybrid of hyperbolic and Euclidean geometry and its relation to topological surfaces. This study of *complex projective structures* draws upon ideas from many areas of mathematics including geometry, topology, complex analysis, group theory, and moduli theory. My postdoctoral work is in the area of *geometric and topological data analysis*, and is an attempt to import recent ideas from geometry and topology to aid in pattern classification problems such as facial recognition, human action and gesture classification and anomaly detection. I am particularly interested in the variation of geometric structures on a topological manifold, discovery of geometric structure in massive data sets, and computation and optimization of such structures for the purpose of visualization and information extraction.

Geometry and Topology. Topology refers to the properties of an object that are preserved through bending, stretching or twisting. For example, a sock and a cowboy hat have the same topology because they both can be flattened out onto a plane without tearing or folding either one. This is great for packing a suitcase! Topology is associated to very basic properties, and is typically used to distinguish objects at a fundamental level. Geometry is more rigid, since it refers to the properties of an object that are preserved by a group of transformations of the object. Parameter spaces of geometric structures on a topological space provide an interesting way to quantify the relationships which exist between these two ideas. For a given topological object X, a point in such a parameter space represents a geometric structure on X. A path in the parameter space represents a process that continuously deforms one geometric structure on X into another. The coordinates of the space are called parameters since they measure the degree to which one can deform the geometry without perturbing the underlying topology. Next, we consider the parameter space of complex projective structures.

Let $\mathcal{G}(S,\rho)$ be a graph whose vertices are complex projective structures with holonomy ρ and whose edges are graftings from one vertex to another. If ρ is quasi-Fuchsian, a theorem of Goldman [4] implies that $\mathcal{G}(S,\rho)$ is connected. If $\rho(\pi_1(S))$ is a Schottky group, Baba [1] has shown that $\mathcal{G}(S,\rho)$ is connected. We show in [9] that if $\rho(\pi_1(S))$ is a Schottky group, then $\pi_1(\mathcal{G}(S,\rho))$ is not finitely generated and there are an infinite number of (standard) projective structures which can be *grafted* to a common structure. A recent preprint of Baba implies that in the Schottky case, $\mathcal{G}(S,\rho)$ is connected. This, plus our connectivity result suggests a rich *calculus* of grafting in this Schottky case and our computational methods provide a starting point for this new area of research. Furthermore, Kapovich [5] has generalized Goldman's result to all higher dimensions, and a generalization of this to the Schottky case remains open. The idea of our main theorem which implies these connectivity results is suggest in the figure below.

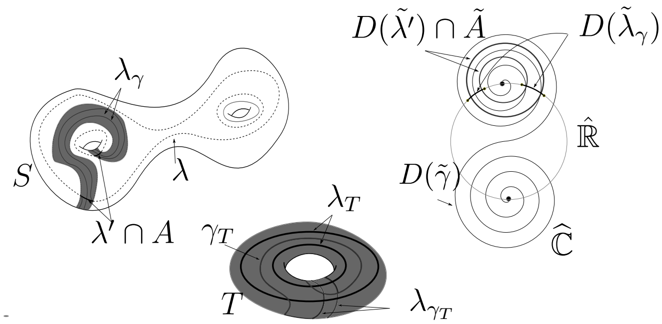

FIGURE 1. Grafting S along a left-spiraling curve γ, the developed image and the Hopf torus.

Future Projects. The construction of computer based models for elementary parameter spaces forms a class of interesting mathematics/computer science interdisciplinary research projects. For example, the machine learning community has recently begun to use parameter spaces to classify patterns in data [7]. Little is known, however, concerning how different choices of metric on the underlying parameter space affect classification rates. A rigorous comparison of such metrics would provide mathematical justification for the use of parameter spaces in this context. Additionally, visualization and computation of geometric structures is an active area of research [11]. Our theoretical results can be implemented to produce computer aided computation and visualization of complex projective structures. Specifically, the process of grafting can be seen as an animation whose frames capture the effect of attaching Euclidean tubes of varying widths to the hyperbolic structure. Students who have knowledge of graph theory, abstract algebra or topology could begin research in this area immediately. Another related project would use Teichmüller theory to classify digital images, extending the work of [3].

Recent results [1], [8] concerning the parameter space of complex projective structures show that these geometric structures can be described in very simple topological terms. These results also provide simple combinatorial tools that make navigation in this space a problem appropriate for undergraduate students. The computation of paths and loops in this space would provide geometric information, and could be approached from several directions such as graph theory and combinatorics. Visualization of these structures is important to understanding them, and computer generated models can be developed in terms of their newfound topological simplicity. A long term goal is to use complex projective structures to aid in pattern classification problems as in [3] and this program of visualization and computation would be a first step in this direction.

Geometric Pattern Analysis. A pattern or data point, can be defined very generally and can refer to a shape, digital image, video or song. For example, pixels can be thought of in terms of matrices and there is a wealth of good problems dealing with linear algebra of digital images. High-resolution data is inherently high-dimensional, so matrix factorization techniques which produce low-rank approximations are used to generate low-dimensional embeddings of the data which simultaneously reduce complexity and preserve important features. These techniques play key roles in the solution to many challenging classification problems such as voice, facial and action recognition as well as anomaly detection [10].

A modern perspective posits that most patterns are generally driven by a few non-linear parameters, creating a need for non-linear techniques such as *manifold learning*, which rely heavily on geometry and topology. Given a set of images, one wishes to define a parameter space for the set whose geometry and/or topology accurately depicts the interrelationships of the images. If this can be done, the geometric structure of the parameter space can be used to classify the images. For example, a set of $k \cdot n$, p-pixel images of n people taken under k different illumination conditions can be mapped to n points on a Grassmann manifold of dimension $k(p-k)$ and the manifold's chordal distance can be used to accurately distinguish the n people [2]. Given an arbitrary data set, the search for a space which parameterizes subsets of high correlation is an exciting and challenging problem involving several different areas of study.

Future Projects. I have several related projects in mind that are appropriate for undergraduate and graduate research in this area. These projects are easy to define and seem approachable to intellectually flexible students with a modest background consisting of elementary calculus and linear algebra. These range in mathematical direction as well as in depth, and I will briefly describe a few below. Students will learn to see digital audio and video as a playground for mathematical ideas such as geometry, topology, linear algebra and calculus.

Complex projective structures are natural tools for data analysis since they are geometric structures on subsets of parameter spaces that are invariant under a group action. I hope to convert theorems of complex projective structures into practical algorithms and implement them for pattern analysis applications. These structures share many similarities to hyperbolic and conformal structures, which have recently been used in applications including medical imaging and shape analysis [3].

A video clip can naturally be thought of as a *data cube* where x and y represent pixel values and t represents time. To *play* the video, we typically display successive (x, y) slices of the video in succession. It is possible, however, to instead display successive (x, t) or (t, y) slices in succession, essentially creating another video from the same data. By focusing on the *set of frames*, each slice determines a subspace of $\mathbb{R}^n$, by taking the span of the set. Then a video naturally determines a point on several Grassmannian manifolds. We have shown [9] that algorithms on such manifolds can perform better than the same algorithm in the classical setting of vector spaces, and other algorithms are likely to benefit from such a generalization.

Another set of projects involves the parameter space of musical chords. It has been shown that the parameter space of musical chords is 3-dimensional and has a nice geometric and topological structure. One interesting project would be to create an interactive

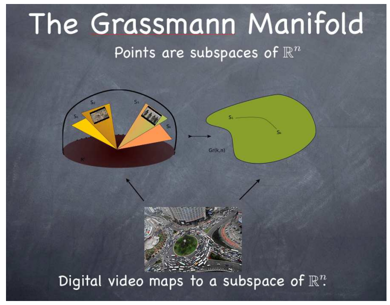

FIGURE 2. The Grassmannian as a parameter space for the set of digital videos.

3-dimensional model of the parameter space in which one could *hear* different points of the parameter space. Another project is to model a set of songs in the parameter space, and see if similarity in the parameter space coincides with human perceptions of similarity.

Conclusion. Parameter spaces of geometric structures have long been of intrinsic theoretical interest. Their recent emergence as a tool for solving interdisciplinary classification problems [9], [7] has made them valuable in the applied setting as well. Visualization, computation and optimization in such spaces are concrete tools through which geometry, topology and parameter spaces are applied to solve classification problems.

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